About intuitionistic public announcement logic

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Abstract

Public announcement logic (PAL) is a logic for reasoning about the dynamic of knowledge in a multi-agent system in which public announcements are made. Syntactically, public announcements are modal formulas. Semantically, they correspond to restrictions of models. In [10], Ma *et al.* use the standard toolkit of duality theory in modal logic to define an algebraic semantics for a combination of IPL and PAL into intuitionistic public announcement logic ($IPAL$). In this paper, grounding our approach on relational semantics rather than on algebraic semantics, we give a sound and complete axiomatization of $IPAL$ and we consider a complete sequent calculus for the associated membership problem.

Keywords: Public announcement logic. Intuitionistic propositional logic. Axiomatization/completeness. Decidability/complexity. Sequent calculus.

1 Introduction

Public announcement logic (PAL) is a logic for reasoning about the dynamic of knowledge in a multi-agent system [16]. Syntactically, public announcements are modal formulas. Semantically, they correspond to restrictions of models. There exist multifarious variants of PAL: PAL with arbitrary public announcements [1], PAL with common knowledge [7], etc. In all these variants, the construct $(\cdot \to \cdot)$ is the one of classical propositional logic. In [10], Ma *et al.* introduce a variant of PAL in which this construct is the one of intuitionistic propositional logic (IPL). By using the standard toolkit of duality theory in modal logic, they define an algebraic semantics for a combination of IPL and PAL into intuitionistic public announcement logic ($IPAL$). In

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this paper, grounding our approach on relational semantics rather than on algebraic semantics, we provide supplementary results about $IPAL$. Firstly, we give a sound and complete axiomatization of $IPAL$ and we prove its completeness. Secondly, we study the features that, according to Simpson [17], might be expected of any intuitionistic modal logic and we examine whether $IPAL$ possesses them. Thirdly, we propose an alternative semantics for $IPAL$, dealing with stacks of annoucements, following the approach developed in [2] for PAL, and then we derive from this semantics a new sequent calculus for $IPAL$ that is sound and complete. Fourthly, we define a translation of $IPAL$'s formulas into formulas of a multimodal logic in which the construct $(\cdot \rightarrow \cdot)$ is the one of classical propositional logic.

2 Syntax and semantics

Let VAR be a countable set of atomic formulas called variables (denoted p, q, etc). The set of all formulas is inductively defined as follows:

- $\phi ::= p \mid \bot \mid (\phi \vee \psi) \mid (\phi \wedge \psi) \mid (\phi \rightarrow \psi) \mid \Box\phi \mid \Diamond\phi \mid [\phi]\psi \mid \langle\phi\rangle\psi$.

$\bot$, $(\cdot \vee \cdot)$, $(\cdot \wedge \cdot)$ and $(\cdot \rightarrow \cdot)$ are the ordinary constructs of IPL, $\Box\cdot$ ("it is necessary that ...") and $\Diamond\cdot$ ("it is possible that ...") are the alethic constructs of modal logic and $[\cdot]\cdot$ ("if ... then, after announcing it, ...") and $\langle\cdot\rangle\cdot$ ("... and, after announcing it, ...") are the announcement constructs of PAL. The IPL constructs $\neg\cdot$ and $(\cdot \leftrightarrow \cdot)$ are defined as usual.

- $\neg\phi ::= (\phi \rightarrow \bot)$,
- $(\phi \leftrightarrow \psi) ::= ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$.

We adopt the standard rules for omission of the parentheses. Note that, following the line of reasoning suggested by [17, Chapter 3], we have added the new alethic constructs $\Box\cdot$ and $\Diamond\cdot$ and the new announcement constructs $[\cdot]\cdot$ and $\langle\cdot\rangle\cdot$ to the ordinary language of IPL. As proved in Section 7 (see Propositions 7.4 and 7.5), the constructs $\Box\cdot$ and $\Diamond\cdot$ are independent in $IPAL$ but $[\cdot]\cdot$ and $\langle\cdot\rangle\cdot$ are interdefinable. For all formulas ϕ, let ϕ^* be the formula obtained by recursively eliminating the alethic constructs and the announcement constructs occurring in ϕ. For all sets x of formulas, let $\Box x = \{\phi\colon \Box\phi \in x\}$ and $\Diamond x = \{\Diamond\phi\colon \phi \in x\}$. Let the size of a formula ϕ (denoted $size(\phi)$) be the number of occurrences of symbols ϕ contains. The size of a finite sequence $(\phi_1, \ldots, \phi_n)$ of formulas (denoted $size(\phi_1, \ldots, \phi_n)$) is the nonnegative integer defined as follows:

- $size(\phi_1, \ldots, \phi_n) = size(\phi_1) + \ldots + size(\phi_n) + n$.

By ϵ, we will denote the empty sequence of formulas. Obviously, $size(\epsilon) = 0$. A frame is a tuple of the form $\mathcal{F} = (W, \leq, R)$ where W is a nonempty set (denoted x, y, etc), $\leq$ is a partial order on W and R is a binary relation on W. The frame $\mathcal{F} = (W, \leq, R)$ is said to be standard if

- $R^{-1}\circ \leq \subseteq \leq \circ R^{-1}$,
- $R\circ \leq \subseteq \leq \circ R$.

A valuation on a frame $\mathcal{F} = (W, \leq, R)$ is a function $V\colon\ VAR \mapsto 2^W$. The valuation V on the frame $\mathcal{F} = (W, \leq, R)$ is said to be upward closed if

- for all $p \in VAR$ and for all $x \in W$, if $x \in V(p)$ then for all $y \in W$, if $x \leq y$ then $y \in V(p)$.

A model is a tuple of the form $\mathcal{M} = (W, \leq, R, V)$ where $\mathcal{F} = (W, \leq, R)$ is a frame and V is a valuation on $\mathcal{F}$. We shall say that the model $\mathcal{M} = (W, \leq, R, V)$ is standard if the frame $\mathcal{F} = (W, \leq, R)$ is standard. The model $\mathcal{M} = (W, \leq, R, V)$ is said to be upward closed if the valuation V on the frame $\mathcal{F} = (W, \leq, R)$ is upward closed. The satisfiability relation between a model $\mathcal{M} = (W, \leq, R, V)$, an element $x \in W$ and a formula ϕ (denoted $\mathcal{M}, x \models \phi$) is inductively defined as follows:

- $\mathcal{M}, x \models p$ iff $x \in V(p)$,
- $\mathcal{M}, x \not\models \bot$,
- $\mathcal{M}, x \models \phi \vee \psi$ iff either $\mathcal{M}, x \models \phi$, or $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \phi \wedge \psi$ iff $\mathcal{M}, x \models \phi$ and $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \phi \to \psi$ iff for all $y \in W$, if $x \leq y$ and $\mathcal{M}, y \models \phi$ then $\mathcal{M}, y \models \psi$,
- $\mathcal{M}, x \models \Box\phi$ iff for all $y, z \in W$, if $x \leq y$ and yRz then $\mathcal{M}, z \models \phi$,
- $\mathcal{M}, x \models \Diamond\phi$ iff there exists $y \in W$ such that xRy and $\mathcal{M}, y \models \phi$,
- $\mathcal{M}, x \models [\phi]\psi$ iff for all $y \in W$, if $x \leq y$ and $\mathcal{M}, y \models \phi$ then $\mathcal{M}_{|\phi}, y \models \psi$,
- $\mathcal{M}, x \models \langle\phi\rangle\psi$ iff $\mathcal{M}, x \models \phi$ and $\mathcal{M}_{|\phi}, x \models \psi$.

In the above definition, $\mathcal{M}_{|\phi} = (W_{|\phi}, \leq_{|\phi}, R_{|\phi}, V_{|\phi})$ is the model such that $W_{|\phi} = \{x \in W\colon\ \mathcal{M}, x \models \phi\}$, $\leq_{|\phi} = \leq \cap (W_{|\phi} \times W_{|\phi})$, $R_{|\phi} = R \cap (W_{|\phi} \times W_{|\phi})$ and for all $p \in VAR$, $V_{|\phi}(p) = V(p) \cap W_{|\phi}$. Notice that the clauses concerning the modal constructs $\Box\cdot$ and $[\cdot]\cdot$ imitate the clauses for the quantifier $\forall$ in first-order intuitionistic logic whereas the clauses concerning $\Diamond\cdot$ and $\langle\cdot\rangle\cdot$ imitate the clauses for $\exists$. See [6, Lemma 5.3.2] for details. Obviously, in any model $\mathcal{M} = (W, \leq, R, V)$,

- $\mathcal{M}, x \models \neg\phi$ iff for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \not\models \phi$,
- $\mathcal{M}, x \models \phi \leftrightarrow \psi$ iff for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \phi$ iff $\mathcal{M}, y \models \psi$.

Note that if $\mathcal{M}$ is upward closed then $\mathcal{M}_{|\phi}$ is upward closed too. The next lemma states that the set of elements satisfying a formula in an upward closed standard model is upward closed too.

Lemma 2.1 *Let ϕ be a formula. For all upward closed standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, if $\mathcal{M}, x \models \phi$ then $\mathcal{M}_{|\phi}$ is upward closed standard and for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \phi$.*

A formula ϕ is said to be globally satisfied in a model $\mathcal{M} = (W, \leq, R, V)$ (denoted $\mathcal{M} \models \phi$) if for all $x \in W$, $\mathcal{M}, x \models \phi$. The following Lemma will be used in Section 7.

Lemma 2.2 *Let ϕ be a formula. Let $\mathcal{M} = (W, \leq, R, V)$ be a model such that*

$\leq$ *is the identity relation on* W. *If* $\phi \in PAL$ *then* $\mathcal{M} \models \phi$.

There are several reasons for being interested in upward closed standard models. Following the usual paradigm for IPL saying that facts should persist in a model as we ascend its partial order, the fact that xRy in a model $\mathcal{M} = (W, \leq, R, V)$ should persist too. Hence, the condition of being standard. Similarly, the fact that $x \in V(p)$ in a model $\mathcal{M} = (W, \leq, R, V)$ should persist too. Thus, the condition of being upward closed.

3 Validities

We shall say that a formula ϕ is ucs-valid (denoted $\models_{ucs} \phi$) if for all upward closed standard models $\mathcal{M}$, $\mathcal{M} \models \phi$.

Proposition 3.1 *The following formulas are ucs-valid and the following inference rules are ucs-validity preserving:*

$A1$ *All instances of* IPL,

$A2$ $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

$A3$ $\Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$,

$A4$ $(\Diamond\phi \to \Box\psi) \to \Box(\phi \to \psi)$,

$A5$ $\Diamond(\phi \vee \psi) \to (\Diamond\phi \vee \Diamond\psi)$,

$A6$ $\neg\Diamond\bot$,

$A7$ $[\phi]p \leftrightarrow (\phi \to p)$,

$A8$ $[\phi]\bot \leftrightarrow \neg\phi$,

$A9$ $[\phi](\psi \vee \chi) \leftrightarrow (\phi \to ([\phi]\psi \vee [\phi]\chi))$,

$A10$ $[\phi](\psi \wedge \chi) \leftrightarrow ([\phi]\psi \wedge [\phi]\chi)$,

$A11$ $[\phi](\psi \to \chi) \leftrightarrow ([\phi]\psi \to [\phi]\chi)$,

$A12$ $[\phi]\Box\psi \leftrightarrow (\phi \to \Box[\phi]\psi)$,

$A13$ $[\phi]\Diamond\psi \leftrightarrow (\phi \to \Diamond\langle\phi\rangle\psi)$,

$A14$ $\langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi)$,

$R1$ *from* ϕ *and* $\phi \to \psi$ *infer* ψ,

$R2$ *from* ϕ *infer* $\Box\phi$,

$R3$ *from* $\phi \leftrightarrow \psi$ *infer* $[\chi]\phi \leftrightarrow [\chi]\psi$.

Proof. When restricted to announcement-free formulas, the formulas $A1$–$A6$ and the inference rules $R1$ and $R2$ have been used by Fischer Servi [8] and Simpson [17, Chapter 3] who have considered the intuitionistic analogue IK of modal logic K. The formulas $A7$, $A8$, $A10$, $A12$ and $A13$ have been used by Ma *et al.* [10] as reduction axioms. Hence, leaving to the reader the proof of the proposition for the formulas $A9$ and $A11$ and the inference rule $R3$, we only prove the proposition for the formula $A14$.
Suppose $\not\models_{ucs} \langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi)$. Let $\mathcal{M} = (W, \leq, R, V)$ be an upward closed standard model and $x \in W$ be such that $\mathcal{M}, x \not\models \langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi)$. Hence, either $\mathcal{M}, x \not\models \langle\phi\rangle\psi \to (\phi \wedge [\phi]\psi)$, or $\mathcal{M}, x \not\models (\phi \wedge [\phi]\psi) \to \langle\phi\rangle\psi$. In the former case, let $y \in W$ be such that $x \leq y$, $\mathcal{M}, y \models \langle\phi\rangle\psi$ and $\mathcal{M}, y \not\models \phi \wedge [\phi]\psi$. Thus, $\mathcal{M}, y \models \phi$, $\mathcal{M}_{|\phi}, y \models \psi$ and $\mathcal{M}, y \not\models [\phi]\psi$. Let $z \in W$ be such that $y \leq z$, $\mathcal{M}, z \models \phi$ and $\mathcal{M}_{|\phi}, z \not\models \psi$. Since $\mathcal{M}, y \models \phi$, therefore $y \leq_{|\phi} z$. Since $\mathcal{M}_{|\phi}, y \models \psi$, therefore by Lemma 2.1, $\mathcal{M}_{|\phi}, z \models \psi$: a contradiction. In the latter case, let $y \in W$ be such that $x \leq y$, $\mathcal{M}, y \models \phi \wedge [\phi]\psi$ and $\mathcal{M}, y \not\models \langle\phi\rangle\psi$. Consequently, $\mathcal{M}, y \models \phi$, $\mathcal{M}, y \models [\phi]\psi$ and $\mathcal{M}_{|\phi}, y \not\models \psi$. Hence, $\mathcal{M}_{|\phi}, y \models \psi$: a contradiction. Thus, $\models_{ucs} \langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi)$. $\Box$

Proposition 3.2 *The following formulas are ucs-valid and the following inference rule is ucs-validity preserving:*

$A15$ $[\phi](\psi \to \chi) \to ([\phi]\psi \to [\phi]\chi)$,

$A16$ $[\phi](\psi \to \chi) \to (\langle\phi\rangle\psi \to \langle\phi\rangle\chi)$,

$A17$ $(\langle\phi\rangle\psi \to [\phi]\chi) \to [\phi](\psi \to \chi)$,

$A18$ $\langle\phi\rangle(\psi \vee \chi) \to (\langle\phi\rangle\psi \vee \langle\phi\rangle\chi)$,

$R4$ *from* ϕ *infer* $[\psi]\phi$.

Proof. Left to the reader. □

Proposition 3.3 *The following formulas are ucs-valid:*

$A19$ $[\phi]\top \leftrightarrow \top$,

$A20$ $\langle\phi\rangle\bot \leftrightarrow \bot$,

$A21$ $\langle\phi\rangle\top \leftrightarrow \phi$,

$A22$ $[\phi](\psi\vee\chi) \leftrightarrow (\phi \to \langle\phi\rangle\psi\vee\langle\phi\rangle\chi)$,

$A23$ $\langle\phi\rangle(\psi \vee \chi) \leftrightarrow \langle\phi\rangle\psi \vee \langle\phi\rangle\chi$,

$A24$ $[\phi](\psi \to \chi) \leftrightarrow (\langle\phi\rangle\psi \to \langle\phi\rangle\chi)$,

$A25$ $\langle\phi\rangle(\psi \to \chi) \leftrightarrow \phi \wedge (\langle\phi\rangle\psi \to \langle\phi\rangle\chi)$,

$A26$ $\langle\phi\rangle\Diamond\psi \leftrightarrow \phi \wedge \Diamond\langle\phi\rangle\psi$,

$A27$ $\langle\phi\rangle p \leftrightarrow \phi \wedge p$,

$A28$ $\langle\phi\rangle(\psi \wedge \chi) \leftrightarrow \langle\phi\rangle\psi \wedge \langle\phi\rangle\chi$,

$A29$ $\langle\phi\rangle\Box\psi \leftrightarrow \phi \wedge \Box[\phi]\psi$.

Proof. Left to the reader. □

Note that the set of all ucs-valid formulas is not closed under the inference rule of uniform substitution. For example, the formula $[p]p$ is ucs-valid but its instance $[q \wedge \Diamond\neg q](q \wedge \Diamond\neg q)$ is not globally satisfied in the upward closed standard model $\mathcal{M} = (W, \leq, R, V)$ where $W = \{x, y\}$, $\leq = \{(x, x), (y, y)\}$, $R = \{(x, y)\}$ and $V(q) = \{x\}$. Hence, we should be very careful when applying to $IPAL$ tools and techniques designed for normal modal logic.

4 Axiomatization/completeness

Let $IPAL$ be the least set of formulas containing the formulas $A1$–$A14$ and closed under the inference rules $R1$–$R3$. The soundness of $IPAL$ relative to its relational semantics is straightforward, seeing that

Proposition 4.1 (Soundness) *Let* ϕ *be a formula. If* $\phi \in IPAL$ *then* $\models_{ucs} \phi$.

Proof. By Proposition 3.1. □

Without using the standard toolkit of duality theory in modal logic and the results in [10], the completeness of $IPAL$ relative to its relational semantics is more difficult to establish than its soundness and we defer proving that $IPAL$ is complete with respect to the class of all upward closed standard models till the end of this section. A useful result is the following

Proposition 4.2 *Let* ϕ *be a formula and* ψ *be an announcement-free formula such that* $\phi \leftrightarrow \psi \in IPAL$. *Let* χ *be an announcement-free formula. There exists an announcement-free formula* θ *such that* $[\phi]\chi \leftrightarrow \theta \in IPAL$. *Moreover, if* ψ *and* χ *are* $\Box$*-free (respectively,* $\Diamond$*-free) then* θ *is* $\Box$*-free (respectively,* $\Diamond$*-free).*

Proof. Let FOR be the set of all announcement-free formulas χ such that there exists an announcement-free formula θ such that $[\phi]\chi \leftrightarrow \theta \in IPAL$ and, moreover, if ψ and χ are $\Box$-free (respectively, $\Diamond$-free) then θ is $\Box$-free (respectively, $\Diamond$-free). Proposition 4.2 says that for all announcement-free formulas χ, $\chi \in FOR$. We will demonstrate it by an induction on χ based on the function $size(\cdot)$ defined in Section 2. Let χ be an announcement-free formula such that for all announcement-free formulas μ, if $size(\mu) < size(\chi)$ then $\mu \in FOR$. We demonstrate $\chi \in FOR$. We only consider the case $\chi = \Diamond\mu$.
Note that $size(\mu) < size(\chi)$. Hence, $\mu \in FOR$. Let θ be an announcement-free formula such that $[\phi]\mu \leftrightarrow \theta \in IPAL$. By $A13$, $[\phi]\Diamond\mu \leftrightarrow (\phi \rightarrow \Diamond\langle\phi\rangle\mu) \in IPAL$. Since $\phi \leftrightarrow \psi \in IPAL$, therefore $[\phi]\Diamond\mu \leftrightarrow (\psi \rightarrow \Diamond\langle\phi\rangle\mu) \in IPAL$. By $A14$, $\langle\phi\rangle\mu \leftrightarrow (\phi \wedge [\phi]\mu) \in IPAL$. Since $\phi \leftrightarrow \psi \in IPAL$ and $[\phi]\mu \leftrightarrow \theta \in IPAL$, therefore $\langle\phi\rangle\mu \leftrightarrow (\psi \wedge \theta) \in IPAL$. Thus, $\Diamond\langle\phi\rangle\mu \leftrightarrow \Diamond(\psi \wedge \theta) \in IPAL$. Since $[\phi]\Diamond\mu \leftrightarrow (\psi \rightarrow \Diamond\langle\phi\rangle\mu) \in IPAL$, therefore $[\phi]\Diamond\mu \leftrightarrow (\psi \rightarrow \Diamond(\psi\wedge\theta)) \in IPAL$. $\Box$

From Proposition 4.2, it follows that

Proposition 4.3 *For all formulas ϕ, there exists an announcement-free formula ψ such that $\phi \leftrightarrow \psi \in IPAL$. Moreover, if ϕ is $\Box$-free (respectively, $\Diamond$-free) then ψ is $\Box$-free (respectively, $\Diamond$-free).*

Proof. Let FOR be the set of all formulas ϕ such that there exists an announcement-free formula ψ such that $\phi \leftrightarrow \psi \in IPAL$ and, moreover, if ϕ is $\Box$-free (respectively, $\Diamond$-free) then ψ is $\Box$-free (respectively, $\Diamond$-free). Proposition 4.3 says that for all formulas ϕ, $\phi \in FOR$. We will demonstrate it by an induction on ϕ based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all announcement-free formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in FOR$. We demonstrate $\phi \in FOR$. We only consider the case $\phi = [\psi]\chi$.
Note that $size(\psi) < size(\phi)$ and $size(\chi) < size(\phi)$. Hence, $\psi \in FOR$ and $\chi \in FOR$. Let θ be an announcement-free formula such that $\psi \leftrightarrow \theta \in IPAL$ and μ be an announcement-free formula such that $\chi \leftrightarrow \mu \in IPAL$. By $R3$, $[\psi]\chi \leftrightarrow [\psi]\mu \in IPAL$. Let ν be an announcement-free formula such that $[\psi]\mu \leftrightarrow \nu \in IPAL$. Such ν exists by Proposition 4.2 because $\psi \leftrightarrow \theta \in IPAL$. Since $[\psi]\chi \leftrightarrow [\psi]\mu \in IPAL$, therefore $[\psi]\chi \leftrightarrow \nu \in IPAL$. $\Box$

Now, we are ready for the proof of the completeness of $IPAL$ relative to its relational semantics.

Proposition 4.4 (Completeness) *Let ϕ be a formula. If $\models_{ucs} \phi$ then $\phi \in IPAL$.*

Proof. Suppose $\models_{ucs} \phi$ and $\phi \notin IPAL$. Let ψ be an announcement-free formula such that $\phi \leftrightarrow \psi \in IPAL$. Such formula exists by Proposition 4.3. Since $\phi \notin IPAL$, therefore $\psi \notin IPAL$. By the Canonical Model Construction described in [17, Chapter 3], $\not\models_{ucs} \psi$. Since $\phi \leftrightarrow \psi \in IPAL$, therefore by Proposition 4.1, $\models_{ucs} \phi \leftrightarrow \psi$. Since $\not\models_{ucs} \psi$, therefore $\not\models_{ucs} \phi$: a contradiction. Hence, if $\models_{ucs} \phi$ then $\phi \in IPAL$. $\Box$

In the definition of $IPAL$, we did not use the formulas $A15$–$A29$ and the inference rule $R4$ considered in Propositions 3.2 and 3.3. Why not? The reason is that neither the formulas $A15$–$A29$ nor the inference rule $R4$ are used in the proof of Propositions 4.2 and 4.3. Moreover,

Proposition 4.5 *The formulas $A15$–$A29$ are in $IPAL$ and the inference rule $R4$ is admissible in $IPAL$.*

Proof. By Propositions 3.2, 3.3, 4.1 and 4.4. □

5 Canonical model

Let L be an extension of $IPAL$, i.e. L is a set of formulas containing the formulas $A1$–$A14$ and closed under the inference rules $R1$–$R3$. For all sets x, y of formulas, y is said to be an L-consequence of x (denoted $x \vdash_L y$) if there exists nonnegative integers m, n and there exists formulas $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n$ such that $\phi_1, \ldots, \phi_m \in x$, $\psi_1, \ldots, \psi_n \in y$ and $\phi_1 \wedge \ldots \wedge \phi_m \rightarrow \psi_1 \vee \ldots \psi_n \in L$. In this definition, if $m = 0$ then we will consider that $\phi_1 \wedge \ldots \wedge \phi_m$ is equal to $\top$ and if $n = 0$ then we will consider that $\psi_1 \vee \ldots \psi_n$ is equal to $\bot$. In the sequel, we will always assume that $\emptyset \nvdash_L \emptyset$, i.e. we will always assume that $\top \rightarrow \bot \notin L$. We shall say that a set x of formulas is L-prime if the following conditions hold:

- for all formulas ϕ, if $x \vdash_L \{\phi\}$ then $\phi \in x$,
- $x \nvdash_L \{\bot\}$,
- for all formulas ϕ, ψ, if $\phi \vee \psi \in x$ then either $\phi \in x$, or $\psi \in x$.

Lemma 5.1 (Prime Lemma) *For all sets x, y of formulas, if $x \nvdash_L y$ then there exists an L-prime set x' of formulas such that $x \subseteq x'$ and $x' \nvdash_L y$.*

Since $\emptyset \nvdash_L \emptyset$, therefore the set of all L-prime sets of formulas is nonempty. L's Canonical Model is the tuple $\mathcal{M}_c = (W_c, \leq_c, R_c, V_c)$ where W_c is the set of all L-prime sets of formulas, $\leq_c$ is the partial order on W_c defined by $x \leq_c y$ iff $x \subseteq y$, R_c is the binary relation on W_c defined by xR_cy iff $\Box x \subseteq y$ and $\Diamond y \subseteq x$ and V_c: $VAR \mapsto 2^{W_c}$ is the function defined by $x \in V_c(p)$ iff $p \in x$.

Lemma 5.2 *The model $\mathcal{M}_c$ is upward closed standard.*

Lemma 5.3 (Restricted Truth Lemma) *Let ϕ be an announcement-free formula. For all L-prime sets x of formulas, the following conditions are equivalent: (i) $\mathcal{M}_c, x \models \phi$, (ii) $\phi \in x$.*

Lemma 5.4 (Truth Lemma) *Let ϕ be a formula. For all L-prime sets x of formulas, the following conditions are equivalent: (i) $\mathcal{M}_c, x \models \phi$, (ii) $\phi \in x$.*

In Section 7, we will consider an extension of $IPAL$ that contains all formulas of the form $\phi \vee \neg\phi$.

Proposition 5.5 *Let L be an extension of $IPAL$ that contains all formulas of the form $\phi \vee \neg\phi$. For all L-primes sets x, y of formulas, if $x \subseteq y$ then $x = y$.*

Proof. Let x, y be L-primes sets of formulas. Suppose $x \subseteq y$ and $x \neq y$. Hence, $y \not\subseteq x$. Let ψ be a formula such that $\psi \in y$ and $\psi \notin x$. Since L is an extension of $IPAL$ that contains all formulas of the form $\phi \vee \neg\phi$, therefore $\psi \vee \neg\psi \in x$. Thus, either $\psi \in x$, or $\neg\psi \in x$. Since $\psi \notin x$, therefore

$\neg\psi \in x$. Since $x \subseteq y$, therefore $\neg\psi \in y$. Since $\psi \in y$, therefore $y \vdash_L \{\bot\}$: a contradiction. Consequently, if $x \subseteq y$ then $x = y$. $\square$

6 Relationship with Ma *et al.* [10]

A formula ϕ is said to be a-valid (denoted $\models_a \phi$) if for all algebraic models $\mathcal{M} = (A, 0_A, 1_A, +_A, \times_A, \Rightarrow_A, l_A, m_A, V)$ (called Fischer Servi models in [10]), $\mid \phi \mid_{\mathcal{M}} = 1_A$.

Proposition 6.1 *Let ϕ be a formula. If $\models_{ucs} \phi$ then $\models_a \phi$.*

Proof. Suppose $\models_{ucs} \phi$ and $\not\models_a \phi$. By Proposition 4.4, $\phi \in IPAL$. Since the formulas considered in Proposition 3.1 are a-valid and the inference rules considered in Proposition 3.1 are a-validity preserving, therefore $\models_a \phi$: a contradiction. $\square$

Proposition 6.2 *Let ϕ be a formula. If $\models_a \phi$ then $\models_{ucs} \phi$.*

Proof. Suppose $\models_a \phi$ and $\not\models_{ucs} \phi$. By [10], ϕ is derivable from the axioms and the inference rules considered in [10, Section 4.1]. Obviously, these axioms are standard-valid and these inference rules are standard-validity preserving. Hence, $\models_{ucs} \phi$: a contradiction. $\square$

Let $IPAL'$ be the least set of formulas containing the formulas $A1$–$A8$, $A10$, $A13$ and $A19$–$A29$ and closed under the inference rules $R1$ and $R2$. The deducibility relation between a finite set X of variables and a formula ϕ (denoted $X \triangleright \phi$) is inductively defined as follows:

- $X \triangleright p$ iff $p \in X$,
- $X \not\triangleright \bot$,
- $X \triangleright \phi \vee \psi$ iff either $X \triangleright \phi$, or $X \triangleright \psi$,
- $X \triangleright \phi \wedge \psi$ iff $X \triangleright \phi$ and $X \triangleright \psi$,
- $X \triangleright \phi \to \psi$ iff if $X \triangleright \phi$ then $X \triangleright \psi$,
- $X \triangleright \Box\phi$ iff $X \triangleright \phi$,
- $X \triangleright \Diamond\phi$ iff $X \triangleright \phi$,
- $X \triangleright [\phi]\psi$ iff if $X \triangleright \phi$ then $X \triangleright \psi^*$,
- $X \triangleright \langle\phi\rangle\psi$ iff $X \triangleright \phi$ and $X \triangleright \psi^*$.

Note that the axioms and the inference rules considered in [10, Section 4.1] do not explicitly contain the inference rule $R3$. Hence, they are those of $IPAL'$. We believe that this absence of the inference rule $R3$ is only a careless mistake, seeing that

Lemma 6.3 *Let X be a finite set of variables and ϕ be a formula. If $\phi \in IPAL'$ then $X \triangleright \phi$.*

Lemma 6.4 *Let X be a finite set of variables. If $p \in X$, $q \notin X$ and $r \in X$ then $X \triangleright \langle p\rangle\langle q\rangle r$ and $X \not\triangleright \langle\langle p\rangle q\rangle r$.*

Proposition 6.5 (i) *$\langle p\rangle\langle q\rangle r \to \langle\langle p\rangle q\rangle r \in IPAL$.*

(ii) *$\langle p\rangle\langle q\rangle r \to \langle\langle p\rangle q\rangle r \notin IPAL'$.*

Proof. (i) It suffices to use the completeness of $IPAL$ (Proposition 4.4) and the fact that $\langle p\rangle\langle q\rangle r \to \langle\langle p\rangle q\rangle r$ is ucs-valid.
(ii) By Lemmas 6.3 and 6.4. $\square$

7 Other properties of $IPAL$

In [17], Simpson discusses what it means to combine IPL and modal logic into intuitionistic modal logic (IML) and isolates features that might be expected of an IML. In the following proposition, we examine whether $IPAL$ complains with Simpson's requirements.

Proposition 7.1 (i) *$IPAL$ is conservative over IPL.*

(ii) *$IPAL$ contains all instances of IPL.*

(iii) *$IPAL$ is closed under modus ponens.*

(iv) *The addition of the formulas of the form $\phi \vee \neg\phi$ to $IPAL$ yields PAL.*

(v) *If $\phi \vee \psi \in IPAL$ then either $\phi \in IPAL$, or $\psi \in IPAL$.*

Proof. (i) Let ϕ be a modality-free formula. To prove that $\phi \in IPAL$ iff $\phi \in IPL$, it suffices to use the soundness/completeness of $IPAL$ (Propositions 4.1 and 4.4) and IPL (Theorem 2.43 in [5]).
(ii) By definition, $IPAL$ contains all instances of IPL.
(iii) By definition, $IPAL$ is closed under modus ponens.
(iv) Let $IPAL^+$ be the axiom system consisting of the addition of the formulas of the form $\phi \vee \neg\phi$ to $IPAL$. Suppose $IPAL^+$ does not yield PAL. Hence, $IPAL^+ \neq PAL$. Obviously, $IPAL^+ \subseteq PAL$. Since $IPAL^+ \neq PAL$, therefore $PAL \not\subseteq IPAL^+$. Let ψ be a formula such that $\psi \in PAL$ and $\psi \notin IPAL^+$. Let χ be an announcement-free formula such that $\psi \leftrightarrow \chi \in IPAL$. Such formula exists by Proposition 4.3. Thus, $\psi \leftrightarrow \chi \in IPAL^+$. Since $\psi \notin IPAL^+$, therefore $\chi \notin IPAL^+$. Let $\mathcal{M}_c = (W_c, \leq_c, R_c, V_c)$ be $IPAL^+$'s Canonical Model. Since $\chi \notin IPAL^+$, therefore by Lemmas 5.1 and 5.4, there exists $x \in W_c$ such that $\mathcal{M}_c, x \not\models \chi$. Since $IPAL^+$ is an extension of $IPAL$ that contains all formulas of the form $\phi \vee \neg\phi$, therefore by Proposition 5.5, $\leq_c$ is the identity relation on W_c. Since $\mathcal{M}_c, x \not\models \chi$, therefore by Lemma 2.2, $\chi \notin PAL$. Obviously, $IPAL \subseteq PAL$. Since $\psi \leftrightarrow \chi \in IPAL$, therefore $\psi \leftrightarrow \chi \in PAL$. Since $\chi \notin PAL$, therefore $\psi \notin PAL$: a contradiction. Consequently, $IPAL^+$ yields PAL.
(v) Suppose $\phi \vee \psi \in IPAL$, $\phi \notin IPAL$ and $\psi \notin IPAL$. By Propositions 4.1 and 4.4, $\models_{ucs} \phi \vee \psi$, $\not\models_{ucs} \phi$ and $\not\models_{ucs} \psi$. Let $\mathcal{M}_1 = (W_1, \leq_1, R_1, V_1)$ be an upward closed standard model such that $\mathcal{M}_1 \not\models \phi$ and $\mathcal{M}_2 = (W_2, \leq_2, R_2, V_2)$ be an upward closed standard model such that $\mathcal{M}_2 \not\models \psi$. Let x be a new element and $\mathcal{M} = (W, \leq, R, V)$ be the model where $W = W_1 \cup W_2 \cup \{x\}$, $\leq = \leq_1 \cup \leq_2 \cup (\{x\} \times W_1) \cup (\{x\} \times W_2)$, $R = R_1 \cup R_2$ and for all $p \in VAR$, $V(p) = V_1(p) \cup V_2(p)$. The reader may easily verify that $\mathcal{M}$ is upward closed standard. Moreover, $\mathcal{M}_1$ and $\mathcal{M}_2$ are generated submodel of $\mathcal{M}$. A result similar to Proposition 2.6 in [4] would lead to the conclusion that the global satisfiability relation is invariant under generated submodels. Since $\mathcal{M}_1 \not\models \phi$ and $\mathcal{M}_2 \not\models \psi$, therefore $\mathcal{M} \not\models \phi$ and $\mathcal{M} \not\models \psi$. Since $\models_{ucs} \phi \vee \psi$, therefore $\mathcal{M}, x \models \phi \vee \psi$. Hence, either $\mathcal{M}, x \models \phi$, or $\mathcal{M}, x \models \psi$. In the former case, let $y \in W_1$ be arbitrary. Thus, $x \leq y$. Since $\mathcal{M}, x \models \phi$, therefore by Lemma 2.1, $\mathcal{M}, y \models \phi$. Consequently, $\mathcal{M}_1, y \models \phi$. Since y was arbitrary, therefore $\mathcal{M}_1 \models \phi$:

a contradiction. In the latter case, let $y \in W_2$ be arbitrary. Hence, $x \leq y$. Since $\mathcal{M}, x \models \psi$, therefore by Lemma 2.1, $\mathcal{M}, y \models \psi$. Thus, $\mathcal{M}_2, y \models \psi$. Since y was arbitrary, therefore $\mathcal{M}_2 \models \psi$: a contradiction. Consequently, if $\phi \vee \psi \in IPAL$ then either $\phi \in IPAL$, or $\psi \in IPAL$. $\square$

In [17, Chapter 3], Simpson proves the following

Proposition 7.2 (i) *For all announcement-free formulas ϕ, if $\phi \in IK$ then $\models_{ucs} \phi$.*

(ii) *For all announcement-free formulas ϕ, if $\models_{ucs} \phi$ then $\phi \in IK$.*

(iii) *There exists no $\Box\cdot$-free announcement-free formula ϕ such that $\Box p \leftrightarrow \phi \in IK$.*

(iv) *There exists no $\Diamond\cdot$-free announcement-free formula ϕ such that $\Diamond p \leftrightarrow \phi \in IK$.*

From the soundness/completeness of $IPAL$ (Propositions 4.1 and 4.4) and IK (Items 1 and 2 of Proposition 7.2), we obtain the following

Proposition 7.3 *$IPAL$ is conservative over IK.*

The following propositions characterize a main difference between, on one hand, the modal constructs $\Box\cdot$ and $\Diamond\cdot$ and, on the other hand, $[\cdot]\cdot$ and $\langle\cdot\rangle\cdot$.

Proposition 7.4 (i) *There exists no $\Box\cdot$-free formula ϕ such that $\Box p \leftrightarrow \phi \in IPAL$.*

(ii) *There exists no $\Diamond\cdot$-free formula ϕ such that $\Diamond p \leftrightarrow \phi \in IPAL$.*

Proof. (i) By Proposition 4.3, Item 3 of Proposition 7.2 and Proposition 7.3.
(ii) By Proposition 4.3, Item 4 of Proposition 7.2 and Proposition 7.3. $\square$

Proposition 7.5 (i) *$[\phi]\psi \leftrightarrow (\phi \to \langle\phi\rangle\psi) \in IPAL$.*

(ii) *$\langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi) \in IPAL$.*

Proof. (i) Suppose $[\phi]\psi \leftrightarrow (\phi \to \langle\phi\rangle\psi) \not\in IPAL$. By Proposition 4.4, $\not\models_{ucs}$ $[\phi]\psi \leftrightarrow (\phi \to \langle\phi\rangle\psi)$. Let $\mathcal{M} = (W, \leq, R, V)$ be an upward closed standard model and $x \in W$ be such that $\mathcal{M}, x \not\models [\phi]\psi \leftrightarrow (\phi \to \langle\phi\rangle\psi)$. Hence, either $\mathcal{M}, x \not\models [\phi]\psi \to (\phi \to \langle\phi\rangle\psi)$, or $\mathcal{M}, x \not\models (\phi \to \langle\phi\rangle\psi) \to [\phi]\psi$. In the former case, let $y \in W$ be such that $x \leq y$, $\mathcal{M}, y \models [\phi]\psi$ and $\mathcal{M}, y \not\models \phi \to \langle\phi\rangle\psi$. Let $z \in W$ be such that $y \leq z$, $\mathcal{M}, z \models \phi$ and $\mathcal{M}, z \not\models \langle\phi\rangle\psi$. Thus, $\mathcal{M}_{|\phi}, z \not\models \psi$. Since $y \leq z$ and $\mathcal{M}, z \models \phi$, therefore $\mathcal{M}, y \not\models [\phi]\psi$: a contradiction. In the latter case, let $y \in W$ be such that $x \leq y$, $\mathcal{M}, y \models \phi \to \langle\phi\rangle\psi$ and $\mathcal{M}, y \not\models [\phi]\psi$. Let $z \in W$ be such that $y \leq z$, $\mathcal{M}, z \models \phi$ and $\mathcal{M}_{|\phi}, z \not\models \psi$. Consequently, $\mathcal{M}, z \not\models \langle\phi\rangle\psi$. Since $y \leq z$ and $\mathcal{M}, z \models \phi$, therefore $\mathcal{M}, y \not\models \phi \to \langle\phi\rangle\psi$: a contradiction. Hence, $[\phi]\psi \leftrightarrow (\phi \to \langle\phi\rangle\psi) \in IPAL$.
(ii) By definition, $\langle\phi\rangle\psi \leftrightarrow (\phi \wedge [\phi]\psi) \in IPAL$. $\square$

8 An alternative semantics

A proof-theoretical analysis of PAL has been proposed in [11] in terms of a sequent calculus following the approach of [13]. Unfortunately, this sequent calculus is not complete as it cannot prove the valid formula $[p \wedge p]q \leftrightarrow [p]q$. For details, see [2] where an alternative semantics for PAL and a sequent calculus with labels that were based on a specific management of a stack of anoucements have been proposed. A similar alternative semantics for $IPAL$ can be proposed too. Its definition necessitates the satisfiability relation between a model $\mathcal{M} = (W, \leq, R, V)$, an element $x \in W$, a finite sequence $\varphi = (\phi_1, \ldots, \phi_n)$ of formulas and a formula ϕ (denoted $\mathcal{M}, x, (\varphi) \Vdash \phi$) inductively defined as follows:

- $\mathcal{M}, x, \epsilon \Vdash p$ iff $x \in V(p)$,
- $\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash p$ iff $\mathcal{M}, x, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, x, (\varphi) \Vdash p$,
- $\mathcal{M}, x, (\varphi) \nVdash \bot$,
- $\mathcal{M}, x, (\varphi) \Vdash \phi \vee \psi$ iff either $\mathcal{M}, x, (\varphi) \Vdash \phi$, or $\mathcal{M}, x, (\varphi) \Vdash \psi$,
- $\mathcal{M}, x, (\varphi) \Vdash \phi \wedge \psi$ iff $\mathcal{M}, x, (\varphi) \Vdash \phi$ and $\mathcal{M}, x, (\varphi) \Vdash \psi$,
- $\mathcal{M}, x, \epsilon \Vdash \phi \rightarrow \psi$ iff for all $y \in W$, if $x \leq y$ and $\mathcal{M}, y, \epsilon \Vdash \phi$ then $\mathcal{M}, y, \epsilon \Vdash \psi$,
- $\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash \phi \rightarrow \psi$ iff for all $y \in W$, if $x \leq y$, $\mathcal{M}, y, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, y, (\varphi, \phi_{n+1}) \Vdash \phi$ then $\mathcal{M}, y, (\varphi, \phi_{n+1}) \Vdash \psi$,
- $\mathcal{M}, x, \epsilon \Vdash \Box\phi$ iff for all $y, z \in W$, if $x \leq y$ and yRz then $\mathcal{M}, z, \epsilon \Vdash \phi$,
- $\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash \Box\phi$ iff for all $y, z \in W$, if $x \leq y$, yRz, $\mathcal{M}, y, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, z, (\varphi) \Vdash \phi_{n+1}$ then $\mathcal{M}, z, (\varphi, \phi_{n+1}) \Vdash \phi$,
- $\mathcal{M}, x, \epsilon \Vdash \Diamond\phi$ iff there exists $y \in W$ such that xRy and $\mathcal{M}, y, \epsilon \Vdash \phi$,
- $\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash \Diamond\phi$ iff there exists $y \in W$ such that xRy, $\mathcal{M}, y, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, y, (\varphi, \phi_{n+1}) \Vdash \phi$,
- $\mathcal{M}, x, \epsilon \Vdash [\phi]\psi$ iff for all $y \in W$, if $x \leq y$ and $\mathcal{M}, y, \epsilon \Vdash \phi$ then $\mathcal{M}, y, (\phi) \Vdash \psi$,
- $\mathcal{M}, x, (\varphi, \phi_{n+1}) \Vdash [\phi]\psi$ iff for all $y \in W$, if $x \leq y$, $\mathcal{M}, y, (\varphi) \Vdash \phi_{n+1}$ and $\mathcal{M}, y, (\varphi, \phi_{n+1}) \Vdash \phi$ then $\mathcal{M}, y, (\varphi, \phi_{n+1}, \phi) \Vdash \psi$,
- $\mathcal{M}, x, (\varphi) \Vdash \langle\phi\rangle\psi$ iff $\mathcal{M}, x, (\varphi) \Vdash \phi$ and $\mathcal{M}, x, (\varphi, \phi) \Vdash \psi$.

The reader may easily verify that the above definition of $\mathcal{M}, x, (\varphi) \Vdash \phi$ is correct decreasing on $size(\varphi, \phi)$. A similar stack-based semantics has been proposed by Balbiani *et al.* [2] within the context of PAL. The main difference with the semantics proposed by [11] lies in our interpretation of $\Box$-based formulas.

Lemma 8.1 *Let $(\phi_1, \ldots, \phi_n)$ be a sequence of formulas and ϕ be a formula. For all models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, the following conditions are equivalent: (i) $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\phi$, (ii) if $\mathcal{M}, x, \epsilon \Vdash \phi_1$, $\ldots$, $\mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$ then $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \phi$.*

9 A labelled sequent calculus

Now, we present a sequent calculus for $IPAL$ that is derived from the stack-based semantics given in the previous section. We propose a labelled calculus

$$\dfrac{}{x(\epsilon):p,\ \Gamma\vdash\Delta,\ x(\epsilon):p}\,\text{ax} \qquad \dfrac{}{x(\epsilon):\bot,\ \Gamma\vdash\Delta}\,\text{L}\bot$$

$$\dfrac{x(\varphi):\phi,\ x(\varphi):p,\ \Gamma\vdash\Delta}{x(\varphi,\phi):p,\ \Gamma\vdash\Delta}\,\text{Lp} \qquad \dfrac{\Gamma\vdash\Delta,\ x(\varphi):\phi \quad \Gamma\vdash\Delta,\ x(\varphi):p}{\Gamma\vdash\Delta,\ x(\varphi,\phi):p}\,\text{Rp}$$

$$\dfrac{x(\varphi):\phi,\ x(\varphi):\psi,\ \Gamma\vdash\Delta}{x(\varphi):\phi\wedge\psi,\ \Gamma\vdash\Delta}\,\text{L}\wedge \qquad \dfrac{\Gamma\vdash\Delta,\ x(\varphi):\phi \quad \Gamma\vdash\Delta,\ x(\varphi):\psi}{\Gamma\vdash\Delta,\ x(\varphi):\phi\wedge\psi}\,\text{R}\wedge$$

$$\dfrac{x(\varphi):\phi,\ \Gamma\vdash\Delta \quad x(\varphi):\psi,\ \Gamma\vdash\Delta}{x(\varphi):\phi\vee\psi,\ \Gamma\vdash\Delta}\,\text{L}\vee$$

$$\dfrac{\Gamma\vdash\Delta, x(\varphi):\phi}{\Gamma\vdash\Delta, x(\varphi):\phi\vee\psi}\,\text{R}\vee^1 \qquad \dfrac{\Gamma\vdash\Delta, x(\varphi):\psi}{\Gamma\vdash\Delta, x(\varphi):\phi\vee\psi}\,\text{R}\vee^2$$

$$\dfrac{x\le y,\Gamma\vdash y(\epsilon):\phi \quad x\le y,\Gamma,y(\epsilon):\psi\vdash\Delta}{x(\epsilon):\phi\to\psi,\Gamma\vdash\Delta}\,\text{L}{\to}^{\epsilon} \qquad \dfrac{\Gamma,x\le y,y(\epsilon):\phi\vdash\Delta,\ y(\epsilon):\psi}{\Gamma\vdash\Delta,\ x(\epsilon):\phi\to\psi}\,\text{R}{\to}^{\epsilon}$$

$$\dfrac{\Gamma,\ x\le y,\ y(\varphi):\phi_{n+1},\vdash y(\varphi,\phi_{n+1}):\phi \quad \Gamma,\ x\le y,\ y(\varphi,\phi_{n+1}):\psi\vdash\Delta}{\Gamma,\ x(\varphi,\phi_{n+1}):\phi\to\psi\vdash\Delta}\,\text{L}{\to}^{\varphi}$$

$$\dfrac{\Gamma,\ x\le y, y(\varphi):\phi_{n+1},\ y(\varphi,\phi_{n+1}):\phi\vdash\Delta,\ y(\varphi,\phi_{n+1}):\psi}{\Gamma\vdash\Delta,\ x(\varphi,\phi_{n+1}):\phi\to\psi}\,\text{R}{\to}^{\varphi}$$

Fig. 1. Inference rules for IPAL - intuitionistic rules.

in which labels are defined for capturing the semantics inside the sequent calculus. This approach based on labels is a uniform approach for designing calculi in various logics like modal or intuitionistic logics [13,17] from Kripke-style semantics. We want to emphasize that starting from our stack-based semantics is central here because the similar semantics proposed for PAL allowed us to propose a new labelled calculus for PAL that corrected the deficiency about completeness of an existing labelled sequent calculus [11]. Therefore we propose a sound and complete calculus with sequents that are with multiconclusions, and with distinguished rules for dealing with empty and non-empty stacks of announcements. Let Var be a countable set of variables (denoted x, y, etc). The sequents are pairs of finite sets of expressions either of the form $x(\varphi):\phi$ read "state x satisfies ϕ with respect to the sequence (φ)", or of the form xRy read "state x is related to state y by means of R". The sequent $\Gamma\vdash\Delta$ means that the conjunction of the expressions in Γ implies the disjunction of the expressions in Δ. Provability is defined as usual: formula ϕ is provable iff the sequent $\vdash x(\epsilon):\phi$ is derivable from the inference rules of the calculus presented in Figures 1 and 2. Let $\mathcal{M}=(W,R,V)$ be a model and $f:Var\mapsto W$. Sequents are pairs of finite sets of expressions either of the form $x(\phi_1,\ldots,\phi_n):\phi$, or of the form xRy. We define the property "$\mathcal{M}$ and f satisfy the expression exp" (denoted $\mathcal{M},f\Vdash exp$) as follows:

- $\mathcal{M},f\Vdash x(\phi_1,\ldots,\phi_n):\phi$ iff $\mathcal{M},f(x),(\phi_1,\ldots,\phi_n)\Vdash\phi$,
- $\mathcal{M},f\Vdash xRy$ iff $f(x)Rf(y)$.

$$\frac{x(\epsilon) : \Box\phi,\ x \leq y, yRz,\ z(\epsilon) : \phi,\ \Gamma \vdash \Delta}{x(\epsilon) : \Box\phi,\ x \leq y, yRz, \Gamma \vdash \Delta}\mathrm{L}\Box^{\epsilon} \qquad \frac{x \leq y, yRz,\ \Gamma \vdash \Delta,\ z(\epsilon) : \phi}{\Gamma \vdash \Delta,\ x(\epsilon) : \Box\phi}\mathrm{R}\Box^{\epsilon}$$

$$\frac{x(\varphi, \phi_{n+1}) : \Box\psi,\ x \leq y, yRz,\ z(\varphi, \phi_{n+1}) : \phi,\ \Gamma \vdash \Delta}{x(\varphi, \phi_{n+1}) : \Box\psi,\ x \leq y, yRz,\ z(\varphi) : \phi_{n+1},\ \Gamma \vdash \Delta}\mathrm{L}\Box^{\varphi}$$

$$\frac{x \leq y, yRz,\ z(\varphi) : \phi_{n+1},\ \Gamma \vdash \Delta,\ z(\varphi, \phi_{n+1}) : \phi}{\Gamma \vdash \Delta,\ x(\varphi, \phi_{n+1}) : \Box\phi}\mathrm{R}\Box^{\varphi}$$

$$\frac{y(\epsilon) : \phi,\ xRy,\ \Gamma \vdash \Delta}{x(\epsilon) : \Diamond\phi,\ \Gamma \vdash \Delta}\mathrm{L}\Diamond^{\epsilon} \qquad \frac{\Gamma \vdash \Delta,\ y(\epsilon) : \phi,\ xRy}{\Gamma \vdash \Delta,\ x(\epsilon) : \Diamond\phi}\mathrm{R}\Diamond^{\epsilon}$$

$$\frac{y(\varphi) : \phi,\ xRy,\ y(\varphi, \phi_{n+1}) : \phi,\ \Gamma \vdash \Delta}{x(\varphi, \phi_{n+1}) : \Diamond\phi,\ \Gamma \vdash \Delta}\mathrm{L}\Diamond^{\varphi} \qquad \frac{\Gamma \vdash \Delta,\ y(\varphi) : \phi_{n+1},\ xRy,\ y(\varphi, \phi_{n+1}) : \phi}{\Gamma \vdash \Delta,\ x(\varphi, \phi_{n+1}) : \Diamond\phi}\mathrm{R}\Diamond^{\varphi}$$

$$\frac{\Gamma,\ x \leq y \vdash \Delta,\ y(\epsilon) : \phi \quad y(\phi) : \psi,\ \Gamma \vdash \Delta}{x(\epsilon) : [\phi]\psi,\ \Gamma \vdash \Delta}\mathrm{L}[]^{\epsilon} \qquad \frac{\Gamma,\ x \leq y,\ y(\epsilon) : \phi \vdash \Delta,\ y(\phi) : \psi}{\Gamma \vdash \Delta,\ x(\epsilon) : [\phi]\psi}\mathrm{R}[]^{\epsilon}$$

$$\frac{x \leq y, y(\varphi) : \phi_{n+1},\ \Gamma \vdash \Delta,\ y(\varphi, \phi_{n+1}) : \phi \quad y(\varphi, \phi_{n+1}) : \psi,\ \Gamma \vdash \Delta}{x(\varphi, \phi_{n+1}) : [\phi]\psi,\ \Gamma \vdash \Delta}\mathrm{L}[]^{\varphi}$$

$$\frac{x \leq y, y(\varphi) : \phi_{n+1}, y(\varphi, \phi_{n+1}) : \phi,\ \Gamma \vdash \Delta,\ y(\varphi, \phi_{n+1}, \phi) : \psi}{\Gamma \vdash \Delta,\ x(\varphi, \phi_{n+1}) : [\phi]\psi}\mathrm{R}[]^{\varphi}$$

$$\frac{\Gamma,\ x(\varphi) : \phi,\ x(\varphi, \phi) : \psi\ \vdash \Delta}{\Gamma,\ x(\varphi) : \langle\phi\rangle\psi, \vdash \Delta}\mathrm{L}{<>} \qquad \frac{\Gamma, \vdash \Delta,\ x(\varphi) : \phi \quad \Gamma, \vdash \Delta,\ x(\varphi, \phi) : \psi}{\Gamma, \vdash x(\varphi) : \langle\phi\rangle\psi}\mathrm{R}{<>}$$

Fig. 2. Inference rules for IPAL - modal rules.

We say that a sequent $\Gamma \vdash \Delta$ is valid iff for all models $\mathcal{M} = (W, R, V)$ and for all $f : Var \mapsto W$, if $\mathcal{M}$ and f satisfy every expression in Γ, then $\mathcal{M}$ and f satisfy some expression in Δ.

Proposition 9.1 *Let ϕ be a formula. If ϕ is provable then ϕ is ucs-valid.*

Proof. It suffices to demonstrate that the inference rules considered in Figures 1 and 2 are validity preserving. □

Proposition 9.2 *Let ϕ be a formula. If ϕ is ucs-valid then ϕ is provable.*

Proof. By Proposition 4.4, it suffices to demonstrate that the formulas considered in Proposition 3.1 are provable and the inference rules considered in Proposition 3.1 are provability preserving. □

In Nomura et al. [14], a labelled sequent calculus has been recently given for $IPAL$. It is basically the same as the one for PAL [15] but with, in some rules, restrictions on labelled expressions on the right-hand side of sequents. As this calculus does not use an announcement stack discipline and has such restrictions, it cannot be directly and easily compared with our new calculus. In future work, we will try to compare them with respect, for instance, to proof-search issues and also to explore possible translations between these calculi.

10 Translation into $S4PAL$

By Gödel's Translation, any formula of the IPL's language can be translated into a formula of the $S4$'s language such that the resulting translation is in $S4$ iff the translated formula is in IPL. See [5, Chapter 3] for details. Within the context of $IPAL$, the translation of a formula ϕ (denoted $\tau(\phi)$) is the formula inductively defined as follows:

- $\tau(p) = \blacksquare p$,
- $\tau(\bot) = \bot$,
- $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$,
- $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$,
- $\tau(\phi \to \psi) = \blacksquare(\tau(\phi) \to \tau(\psi))$,
- $\tau(\Box\phi) = \blacksquare\Box\tau(\phi)$,
- $\tau(\Diamond\phi) = \Diamond\tau(\phi)$,
- $\tau([\phi]\psi) = \blacksquare[\tau(\phi)]\tau(\psi)$,
- $\tau(\langle\phi\rangle\psi) = \langle\tau(\phi)\rangle\tau(\psi)$.

The resulting translations belong to the $S4PAL$'s language, i.e. the set of all formulas inductively defined as follows:

- $\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \blacksquare\phi \mid \Box\phi \mid [\phi]\psi$.

In the $S4PAL$'s language, the Boolean constructs $(\cdot \wedge \cdot)$ and $(\cdot \to \cdot)$, the modal constructs $\blacklozenge\cdot$ and $\Diamond\cdot$ and the announcement construct $\langle\cdot\rangle\cdot$ are defined as usual. Moreover, the standard rules for omission of the parentheses are adopted. The formulas of the $S4PAL$'s language are interpreted in models, their $\leq$ binary relations being used to interpret $\blacksquare$-based formulas and their R binary relations being used to interpret $\Box$-based formulas. More precisely, the satisfiability relation between a model $\mathcal{M} = (W, \leq, R, V)$, an element $x \in W$ and a formula ϕ in the $S4PAL$'s language (denoted $\mathcal{M}, x \models \phi$) is inductively defined as follows:

- $\mathcal{M}, x \models p$ iff $x \in V(p)$,
- $\mathcal{M}, x \not\models \bot$,
- $\mathcal{M}, x \models \phi \vee \psi$ iff either $\mathcal{M}, x \models \phi$, or $\mathcal{M}, x \models \psi$,
- $\mathcal{M}, x \models \blacksquare\phi$ iff for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \phi$,
- $\mathcal{M}, x \models \Box\phi$ iff for all $y \in W$, if xRy then $\mathcal{M}, y \models \phi$,
- $\mathcal{M}, x \models [\phi]\psi$ iff if $\mathcal{M}, x \models \phi$ then $\mathcal{M}_{|\phi}, x \models \psi$.

In the above definition, $\mathcal{M}_{|\phi} = (W_{|\phi}, \leq_{|\phi}, R_{|\phi}, V_{|\phi})$ is the model such that $W_{|\phi} = \{x \in W\colon\ \mathcal{M}, x \models \phi\}$, $\leq_{|\phi} = \leq \cap (W_{|\phi} \times W_{|\phi})$, $R_{|\phi} = R \cap (W_{|\phi} \times W_{|\phi})$ and for all $p \in VAR$, $V_{|\phi}(p) = V(p) \cap W_{|\phi}$. Note that if $\mathcal{M}$ is upward closed then $\mathcal{M}_{|\phi}$ is upward closed too. However, there exists a standard model $\mathcal{M} = (W, R, V)$, there exists $x \in W$ and there exists an announcement formula ϕ in the $S4PAL$'s language such that $\mathcal{M}, x \models \phi$ and $\mathcal{M}_{|\phi}$ is not standard. For example, in the standard model $\mathcal{M} = (W, \leq, R, V)$ where $W = \{x, y, z, t, u\}$, $\leq = \{(x,x), (x,t), (y,y), (y,z), (z,z), (t,t), (u,u)\}$, $R = \{(x,y), (t,z), (t,u)\}$ and $V(p) = \{y, z\}$, we have $\mathcal{M}, x \models \Box p$, $\mathcal{M}, y \models \Box p$, $\mathcal{M}, z \models \Box p$, $\mathcal{M}, t \not\models \Box p$ and $\mathcal{M}, u \models \Box p$. Hence, $\mathcal{M}_{|\Box p} = (W_{|\Box p}, \leq_{|\Box p}, R_{|\Box p}, V_{|\Box p})$ where $W_{|\Box p} =$

$\{x, y, z, u\}$, $\leq_{|\square p}= \{(x, x), (y, y), (y, z), (z, z), (u, u)\}$, $R_{|\square p} = \{(x, y)\}$ and $V_{|\square p}(p) = \{y, z\}$ is not standard. Nevertheless, this never happens when the announcement formula ϕ is the resulting translation of a formula in the $IPAL$'s language.

Lemma 10.1 *Let ϕ be a formula in the $IPAL$'s language. For all standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, if $\mathcal{M}, x \models \tau(\phi)$ then $\mathcal{M}_{|\tau(\phi)}$ is standard and for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \tau(\phi)$.*

Lemma 10.2 *Let ϕ be a formula in the $IPAL$'s language. The formula $\tau(\phi) \rightarrow \blacksquare\tau(\phi)$ is s-valid.*

A formula ϕ in the $S4PAL$'s language is said to be globally satisfied in a model $\mathcal{M} = (W, \leq, R, V)$ (denoted $\mathcal{M} \models \phi$) if for all $x \in W$, $\mathcal{M}, x \models \phi$. We shall say that a formula ϕ in the $S4PAL$'s language is s-valid (denoted $\models_s \phi$) if for all standard models $\mathcal{M}$, $\mathcal{M} \models \phi$.

Lemma 10.3 *Let ϕ be a formula in the $IPAL$'s language. For all upward closed standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, the following conditions are equivalent: (i) $\mathcal{M}, x \models \phi$, (ii) $\mathcal{M}, x \models \tau(\phi)$.*

Proposition 10.4 *Let ϕ be a formula in the $IPAL$'s language. The following conditions are equivalent: (i) $\models_{ucs} \phi$, (ii) $\models_s \tau(\phi)$.*

Proof. (i)$\Rightarrow$(ii): By Proposition 4.4, it suffices to demonstrate that the resulting translations of the formulas $A1$–$A14$ are s-valid and that the resulting translations of the inference rules $(R1)$–$(R3)$ are s-validity preserving.
(ii)$\Rightarrow$(i): By Lemma 10.3. $\square$

Obviously, for all formulas ϕ in the $IPAL$'s language, $size(\tau(\phi)) \leq 2 \times size(\phi)$. Nevertheless, seeing that the complexity of the membership problem in the set of all s-valid formulas in $S4PAL$'s language is unknown, Proposition 10.4 does not give us any upper bound on the complexity of the membership problem in the set of all ucs-valid formulas in $IPAL$'s language.

11 Conclusion

In this paper, firstly, we have given a sound and complete axiomatization of $IPAL$ and we have proved its completeness. Secondly, we have studied the features that might be expected of any intuitionistic modal logic and we have examined whether $IPAL$ possesses them. Thirdly, we have proposed an alternative semantics for $IPAL$ and we have designed a new sequent calculus for $IPAL$ that is sound and complete. Fourthly, we have defined a translation of $IPAL$'s formulas into formulas of a multimodal logic in which the construct $(\cdot \rightarrow \cdot)$ is the one of classical propositional logic. Much remains to be done: computability of the membership problem in the set of all ucs-valid formulas in $IPAL$'s language; multi-agent variants with or without positive introspection, negative introspection, common knowledge, distributed knowledge, etc; extension of our framework to intermediate logics.

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References

[1] Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., de Lima, T.: *'Knowable' as 'known after an announcement'.* The Review of Symbolic Logic **1** (2008) 305–334.

[2] Balbiani, P., Demange, V., Galmiche, D.: *A sequent calculus with labels for PAL.* Short paper presented during AiML 2014.

[3] Balbiani, P., van Ditmarsch, H., Herzig, A., de Lima, T.: *Tableaux for public announcement logic.* Journal of Logic and Computation **20** (2010) 55–76.

[4] Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic.* Cambridge University Press (2001).

[5] Chagrov, A., Zakharyaschev, M.: *Modal Logic.* Oxford University Press (1997).

[6] Van Dalen, D.: *Logic and Structure.* Springer (1983).

[7] Van Ditmarsch, H., van der Hoek, W., Kooi, B.: *Dynamic Epistemic Logic.* Springer (2007).

[8] Fischer Servi G.: *Axiomatizations for some intuitionistic modal logics.* Rendiconti del Seminario Matematico Università e Politecnico di Torino **42** (1984) 179–194.

[9] Lutz, C.: *Complexity and succinctness of public announcement logic.* In: Proceedings of the Fifth International Joint Conference on Autonomous Agents and Multiagent Systems. ACM (2007) 137–143.

[10] Ma, M., Palmigiano, A., Sadrzadeh, M.: *Algebraic semantics and model completeness for intuitionistic public announcement logic.* Annals of Pure and Applied Logic **165** (2014) 963–995.

[11] Maffezioli, P., Negri, S.: *A Gentzen-style analysis of public announcement logic.* In Arrazola, X., Ponte, M. (editors): Proceedings of the International Workshop on Logic and Philosophy of Knowledge, Communication and Action. University of the Basque Country Press (2010) 303–323.

[12] Maffezioli, P., Negri, S.: *A proof-theoretical perspective on public announcement logic.* Logic and Philosophy of Science **9** (2011) 49–59.

[13] Negri, S.: *Proof analysis in modal logic.* Journal of Philosophical Logic **34** (2005) 507–544.

[14] Nomura, S., Sano, K., Tojo, S.: *A labelled sequent calculus for intuitionistic public announcement logic.* In Davis, M., Fehnker, A., McIver, A., Voronkov, A. (editors): Logic for Programming, Artificial Intelligence, and Reasoning. Springer (2015) 187–202.

[15] Nomura, S., Sano, K., Tojo, S.: *Revising a sequent calculus for public announcement logic.* In Fenrong, L., Ono, H. (editors): Structural Analysis of Non-classical Logics. Springer (2016) 131-157.

[16] Plaza, J.: *Logics of public communications.* Synthese **158** (2007) 165–179.

[17] Simpson, A.: *The Proof Theory and Semantics of Intuitionistic Modal Logic.* PhD Thesis at the University of Edinburgh (1994).

Annex

Proof of Lemma 2.1: Let FOR be the set of all formulas ϕ such that for all upward closed standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, if $\mathcal{M}, x \models \phi$ then $\mathcal{M}_{|\phi}$ is upward closed standard and for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \phi$. Lemma 2.1 says that for all formulas ϕ, $\phi \in FOR$.

We will demonstrate it by an induction on ϕ based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in FOR$. We demonstrate $\phi \in FOR$. We only consider the case $\phi = \Diamond\psi$. Note that $size(\psi) < size(\phi)$. Hence, $\psi \in FOR$. Let $\mathcal{M} = (W, \leq, R, V)$ be an upward closed standard model and $x \in W$ be such that $\mathcal{M}, x \models \Diamond\psi$.

Let $y, z, t \in W_{|\Diamond\psi}$ be such that $y \leq_{|\Diamond\psi} z$ and $yR_{|\Diamond\psi}t$. We demonstrate there exists $u \in W_{|\Diamond\psi}$ such that $zR_{|\Diamond\psi}u$ and $t \leq_{|\Diamond\psi} u$. Since $y \leq_{|\Diamond\psi} z$ and $yR_{|\Diamond\psi}t$, therefore $y \leq z$ and yRt. Let $u \in W$ be such that zRu and $t \leq u$. Such u exists because $\mathcal{M}$ is standard. Since $t \in W_{|\Diamond\psi}$, therefore $\mathcal{M}, t \models \Diamond\psi$. Hence, there exists $v \in W$ such that tRv and $\mathcal{M}, v \models \psi$. Let $w \in W$ be such that uRw and $v \leq w$. Such w exists because $\mathcal{M}$ is standard and $t \leq u$. Since $\mathcal{M}$ is upward closed standard, $\psi \in FOR$ and $\mathcal{M}, v \models \psi$, therefore $\mathcal{M}, w \models \psi$. Since uRw, therefore $\mathcal{M}, u \models \Diamond\psi$. Thus, $u \in W_{|\Diamond\psi}$. Since $z, t \in W_{|\Diamond\psi}$, zRu and $t \leq u$, therefore $zR_{|\Diamond\psi}u$ and $t \leq_{|\Diamond\psi} u$.

Let $y, z, t \in W_{|\Diamond\psi}$ be such that $yR_{|\Diamond\psi}z$ and $z \leq_{|\Diamond\psi} t$. We demonstrate there exists $u \in W_{|\Diamond\psi}$ such that $y \leq_{|\Diamond\psi} u$ and $uR_{|\Diamond\psi}t$. Since $yR_{|\Diamond\psi}z$ and $z \leq_{|\Diamond\psi} t$, therefore yRz and $z \leq t$. Let $u \in W$ be such that $y \leq u$ and uRt. Such u exists because $\mathcal{M}$ is standard. Since $y \in W_{|\Diamond\psi}$, therefore $\mathcal{M}, y \models \Diamond\psi$. Hence, there exists $v \in W$ such that yRv and $\mathcal{M}, v \models \psi$. Let $w \in W$ be such that uRw and $v \leq w$. Such w exists because $\mathcal{M}$ is standard and $y \leq u$. Since $\mathcal{M}$ is upward closed standard, $\psi \in FOR$ and $\mathcal{M}, v \models \psi$, therefore $\mathcal{M}, w \models \psi$. Since uRw, therefore $\mathcal{M}, u \models \Diamond\psi$. Thus, $u \in W_{|\Diamond\psi}$. Since $y, t \in W_{|\Diamond\psi}$, $y \leq u$ and uRt, therefore $y \leq_{|\Diamond\psi} u$ and $uR_{|\Diamond\psi}t$.

Let $y \in W$ be such that $x \leq y$. We demonstrate $\mathcal{M}, y \models \Diamond\psi$. Since $\mathcal{M}, x \models \Diamond\psi$, therefore there exists $z \in W$ such that xRz and $\mathcal{M}, z \models \psi$. Let $t \in W$ be such that yRt and $z \leq t$. Such t exists because $\mathcal{M}$ is standard and $x \leq y$. Since $\mathcal{M}$ is upward closed standard, $\psi \in FOR$ and $\mathcal{M}, z \models \psi$, therefore $\mathcal{M}, t \models \psi$. Since yRt, therefore $\mathcal{M}, y \models \Diamond\psi$.

Proof of Lemma 2.2: Suppose $\phi \in PAL$. Hence, ϕ is globally PAL-satisfied in $\mathcal{M}$. Since $\leq$ is the identity relation on W, therefore one can demonstrate by an induction on ψ based on the function $size(\cdot)$ defined in Section 2, that for all formulas ψ and for all $x \in W$, $\mathcal{M}, x \models \psi$ iff ψ is PAL-satisfied at x in $\mathcal{M}$. Since ϕ is globally PAL-satisfied in $\mathcal{M}$, therefore $\mathcal{M} \models \phi$.

Proof of Lemma 5.1: The proof is similar to the proof in [17, Chapter 3].

Proof of Lemma 5.2: The proof is similar to the proof in [17, Chapter 3].

Proof of Lemma 5.3: The proof is similar to the proof in [17, Chapter 3].

Proof of Lemma 5.4: By Proposition 4.3, let ψ be an announcement-free formula such that $\phi \leftrightarrow \psi \in IPAL$. Hence, the following conditions are equivalent: (i) $\mathcal{M}_c, x \models \phi$, (ii) $\mathcal{M}_c, x \models \psi$, (iii) $\psi \in x$, (iv) $\phi \in x$. The equivalence between (i) and (ii) follows from Proposition 4.1, Lemma 5.2 and the fact that $\phi \leftrightarrow \psi \in IPAL$. The equivalence between (ii) and (iii) follows from Lemma 5.3. The equivalence between (iii) and (iv) follows from the fact that L is an extension of $IPAL$ and $\phi \leftrightarrow \psi \in IPAL$.

Proof of Lemma 6.3: It suffices to demonstrate that the formulas $A1$–$A14$ are X-deducible and that the inference rules $(R1)$ and $(R2)$ are X-deducibility preserving.

Proof of Lemma 8.1: Let FOR^+ be the set of all nonempty sequences $(\phi_1, \ldots, \phi_n, \phi)$ of formulas such that for all models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\phi$ iff if $\mathcal{M}, x, \epsilon \Vdash \phi_1, \ldots, \mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$ then $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \phi$. Lemma 8.1 says that for all nonempty sequences $(\phi_1, \ldots, \phi_n, \phi)$ of formulas $(\phi_1, \ldots, \phi_n, \phi) \in FOR^+$. We will demonstrate it by an induction on $(\phi_1, \ldots, \phi_n, \phi)$ based on the function $size(\cdot)$ defined in Section 2. Let $(\phi_1, \ldots, \phi_n, \phi)$ be a nonempty sequence of formulas such that for all nonempty sequences $(\phi'_1, \ldots, \phi'_{n'}, \phi')$, if $size(\phi'_1, \ldots, \phi'_{n'}, \phi') < size(\phi_1, \ldots, \phi_n, \phi)$ then $(\phi'_1, \ldots, \phi'_{n'}, \phi') \in FOR^+$. We demonstrate $(\phi_1, \ldots, \phi_n, \phi) \in FOR^+$. We only consider the case $\phi = \Diamond\psi$. Note that for all $i = 1 \ldots n$, $size(\phi_1, \ldots, \phi_{i-1}, \phi_i) < size(\phi_1, \ldots, \phi_n, \phi)$. Moreover, $size(\phi_1, \ldots, \phi_n, \psi) < size(\phi_1, \ldots, \phi_n, \phi)$. Hence, for all $i = 1 \ldots n$, $(\phi_1, \ldots, \phi_{i-1}, \phi_i) \in FOR^+$. Moreover, $(\phi_1, \ldots, \phi_n, \psi) \in FOR^+$. Let $\mathcal{M} = (W, \leq, R, V)$ be a model and let $x \in W$. Leaving the case $n = 0$ to the reader, we assume that $n \geq 1$.

Suppose $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\Diamond\psi$. Suppose $\mathcal{M}, x, \epsilon \Vdash \phi_1, \ldots, \mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$. Since for all $i = 1 \ldots n$, $(\phi_1, \ldots, \phi_{i-1}, \phi_i) \in FOR^+$, therefore for all $i = 1 \ldots n$, $\mathcal{M}, x \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$. Let $y \in W$ be such that for all $i = 1 \ldots n$, $\mathcal{M}, y \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$, xRy and $\mathcal{M}, y \models [\phi_1] \ldots [\phi_n]\psi$. Such y exists because $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\Diamond\psi$ and for all $i = 1 \ldots n$, $\mathcal{M}, x \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$. Since for all $i = 1 \ldots n$, $(\phi_1, \ldots, \phi_{i-1}, \phi_i) \in FOR^+$ and $(\phi_1, \ldots, \phi_n, \psi) \in FOR^+$, therefore for all $i = 1 \ldots n$, $\mathcal{M}, y, (\phi_1, \ldots, \phi_{i-1}) \Vdash \phi_i$ and $\mathcal{M}, y, (\phi_1, \ldots, \phi_n) \Vdash \psi$. Since xRy, therefore $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \Diamond\psi$.

Suppose if $\mathcal{M}, x, \epsilon \Vdash \phi_1, \ldots, \mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$ then $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \Diamond\psi$. Suppose $\mathcal{M}, x \not\models [\phi_1] \ldots [\phi_n]\Diamond\psi$. Hence, for all $i = 1 \ldots n$, $\mathcal{M}, x \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$ and for all $y \in W$, if for all $i = 1 \ldots n$, $\mathcal{M}, y \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$ and xRy then $\mathcal{M}, y \not\models [\phi_1] \ldots [\phi_n]\psi$. Since for all $i = 1 \ldots n$, $(\phi_1, \ldots, \phi_{i-1}, \phi_i) \in FOR^+$, therefore for all $i = 1 \ldots n$, $\mathcal{M}, x, (\phi_1, \ldots, \phi_{i-1}) \Vdash \phi_i$. Since if $\mathcal{M}, x, \epsilon \Vdash \phi_1, \ldots, \mathcal{M}, x, (\phi_1, \ldots, \phi_{n-1}) \Vdash \phi_n$ then $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \Diamond\psi$, therefore $\mathcal{M}, x, (\phi_1, \ldots, \phi_n) \Vdash \Diamond\psi$. Let $y \in W$ be such that for all $i = 1 \ldots n$, $\mathcal{M}, y, (\phi_1, \ldots, \phi_{i-1}) \Vdash \phi_i$, xRy and

$\mathcal{M}, y, (\phi_1, \ldots, \phi_n) \Vdash \psi$. Since for all $i = 1 \ldots n$, $(\phi_1, \ldots, \phi_{i-1}, \phi_i) \in FOR^+$ and $(\phi_1, \ldots, \phi_n, \psi) \in FOR^+$, therefore for all $i = 1 \ldots n$, $\mathcal{M}, y \models [\phi_1] \ldots [\phi_{i-1}]\phi_i$ and $\mathcal{M}, y \models [\phi_1] \ldots [\phi_n]\psi$. Since xRy, therefore $\mathcal{M}, x \models [\phi_1] \ldots [\phi_n]\Diamond\psi$: a contradiction.

Proof of Lemma 10.1: Let FOR be the set of all formulas ϕ in the $IPAL$'s language such that for all standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, if $\mathcal{M}, x \models \tau(\phi)$ then $\mathcal{M}_{|\tau(\phi)}$ is standard and for all $y \in W$, if $x \leq y$ then $\mathcal{M}, y \models \tau(\phi)$. Lemma 10.1 says that for all formulas ϕ in the $IPAL$'s language, $\phi \in FOR$. We will demonstrate it by an induction on ϕ based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in FOR$. We demonstrate $\phi \in FOR$. We only consider the case $\phi = \Diamond\psi$. Note that $size(\psi) < size(\phi)$. Hence, $\psi \in FOR$. Let $\mathcal{M} = (W, \leq, R, V)$ be a standard model and $x \in W$ be such that $\mathcal{M}, x \models \Diamond\tau(\psi)$.
Let $y, z, t \in W_{|\Diamond\tau(\psi)}$ be such that $y \leq_{|\Diamond\tau(\psi)} z$ and $yR_{|\Diamond\tau(\psi)}t$. We demonstrate there exists $u \in W_{|\Diamond\tau(\psi)}$ such that $zR_{|\Diamond\tau(\psi)}u$ and $t \leq_{|\Diamond\tau(\psi)} u$. Since $y \leq_{|\Diamond\tau(\psi)} z$ and $yR_{|\Diamond\tau(\psi)}t$, therefore $y \leq z$ and yRt. Let $u \in W$ be such that zRu and $t \leq u$. Such u exists because $\mathcal{M}$ is standard. Since $t \in W_{|\Diamond\tau(\psi)}$, therefore $\mathcal{M}, t \models \Diamond\tau(\psi)$. Hence, there exists $v \in W$ such that tRv and $\mathcal{M}, v \models \tau(\psi)$. Let $w \in W$ be such that uRw and $v \leq w$. Such w exists because $\mathcal{M}$ is standard and $t \leq u$. Since $\mathcal{M}$ is standard, $\psi \in FOR$ and $\mathcal{M}, v \models \tau(\psi)$, therefore $\mathcal{M}, w \models \tau(\psi)$. Since uRw, therefore $\mathcal{M}, u \models \Diamond\tau(\psi)$. Thus, $u \in W_{|\Diamond\tau(\psi)}$. Since $z, t \in W_{|\Diamond\tau(\psi)}$, zRu and $t \leq u$, therefore $zR_{|\Diamond\tau(\psi)}u$ and $t \leq_{|\Diamond\tau(\psi)} u$.
Let $y, z, t \in W_{|\Diamond\tau(\psi)}$ be such that $yR_{|\Diamond\tau(\psi)}z$ and $z \leq_{|\Diamond\tau(\psi)} t$. We demonstrate there exists $u \in W_{|\Diamond\tau(\psi)}$ such that $y \leq_{|\Diamond\tau(\psi)} u$ and $uR_{|\Diamond\tau(\psi)}t$. Since $yR_{|\Diamond\tau(\psi)}z$ and $z \leq_{|\Diamond\tau(\psi)} t$, therefore yRz and $z \leq t$. Let $u \in W$ be such that $y \leq u$ and uRt. Such u exists because $\mathcal{M}$ is standard. Since $y \in W_{|\Diamond\tau(\psi)}$, therefore $\mathcal{M}, y \models \Diamond\tau(\psi)$. Hence, there exists $v \in W$ such that yRv and $\mathcal{M}, v \models \tau(\psi)$. Let $w \in W$ be such that uRw and $v \leq w$. Such w exists because $\mathcal{M}$ is standard and $y \leq u$. Since $\mathcal{M}$ is standard, $\psi \in FOR$ and $\mathcal{M}, v \models \tau(\psi)$, therefore $\mathcal{M}, w \models \tau(\psi)$. Since uRw, therefore $\mathcal{M}, u \models \Diamond\tau(\psi)$. Thus, $u \in W_{|\Diamond\tau(\psi)}$. Since $y, t \in W_{|\Diamond\tau(\psi)}$, $y \leq u$ and uRt, therefore $y \leq_{|\Diamond\tau(\psi)} u$ and $uR_{|\Diamond\tau(\psi)}t$.
Let $y \in W$ be such that $x \leq y$. We demonstrate $\mathcal{M}, y \models \Diamond\tau(\psi)$. Since $\mathcal{M}, x \models \Diamond\tau(\psi)$, therefore there exists $z \in W$ such that xRz and $\mathcal{M}, z \models \tau(\psi)$. Let $t \in W$ be such that yRt and $z \leq t$. Such t exists because $\mathcal{M}$ is standard and $x \leq y$. Since $\mathcal{M}$ is standard, $\psi \in FOR$ and $\mathcal{M}, z \models \tau(\psi)$, therefore $\mathcal{M}, t \models \tau(\psi)$. Since yRt, therefore $\mathcal{M}, y \models \Diamond\tau(\psi)$.

Proof of Lemma 10.2: Let FOR be the set of all formulas ϕ in the $IPAL$'s language such that the formula $\tau(\phi) \rightarrow \blacksquare\tau(\phi)$ is s-valid. Lemma 10.2 says that for all formulas ϕ in the $IPAL$'s language, $\phi \in FOR$. We will demonstrate it by an induction on ϕ based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in FOR$.

We demonstrate $\phi \in FOR$. We only consider the case $\phi = \langle\psi\rangle\chi$. Note that $size(\psi) < size(\phi)$ and $size(\chi) < size(\phi)$. Hence, $\psi \in FOR$ and $\chi \in FOR$. Thus, the formulas $\tau(\psi) \rightarrow \blacksquare\tau(\psi)$ and $\tau(\chi) \rightarrow \blacksquare\tau(\chi)$ are s-valid. Let us consider the following formulas: (i) $\langle\tau(\psi)\rangle\tau(\chi)$, (ii) $\tau(\psi) \wedge [\tau(\psi)]\tau(\chi)$, (iii) $\blacksquare\tau(\psi) \wedge [\tau(\psi)]\blacksquare\tau(\chi)$, (iv) $\blacksquare\tau(\psi) \wedge (\tau(\psi) \rightarrow \blacksquare[\tau(\psi)]\tau(\chi))$, (v) $\blacksquare\tau(\psi) \wedge \blacksquare[\tau(\psi)]\tau(\chi)$, (vi) $\blacksquare(\tau(\psi) \wedge [\tau(\psi)]\tau(\chi))$, (vii) $\blacksquare\langle\tau(\psi)\rangle\tau(\chi)$. The s-validity of the formula (i)$\rightarrow$(ii) follows from the definition of the satisfiability of formulas in the $S4PAL$'s language. The s-validity of the formula (ii)$\rightarrow$(iii) follows from the s-validity of the formulas $\tau(\psi) \rightarrow \blacksquare\tau(\psi)$ and $\tau(\chi) \rightarrow \blacksquare\tau(\chi)$. The s-validity of the formulas (iii)$\rightarrow$(iv), (iv)$\rightarrow$(v), (v)$\rightarrow$(vi) and (vi)$\rightarrow$(vii) follows from the definition of the satisfiability of formulas in the $S4PAL$'s language.

Proof of Lemma 10.3: Let FOR be the set of all formulas ϕ in the $IPAL$'s language such that for all upward closed standard models $\mathcal{M} = (W, \leq, R, V)$ and for all $x \in W$, $\mathcal{M}, x \models \phi$ iff $\mathcal{M}, x \models \tau(\phi)$. Lemma 10.3 says that for all formulas ϕ in the $IPAL$'s language, $\phi \in FOR$. We will demonstrate it by an induction on ϕ based on the function $size(\cdot)$ defined in Section 2. Let ϕ be a formula such that for all formulas ψ, if $size(\psi) < size(\phi)$ then $\psi \in FOR$. We demonstrate $\phi \in FOR$. We only consider the case $\phi = \langle\psi\rangle\chi$. Note that $size(\psi) < size(\phi)$ and $size(\chi) < size(\phi)$. Hence, $\psi \in FOR$ and $\chi \in FOR$. Let $\mathcal{M} = (W, \leq, R, V)$ be an upward closed standard model and $x \in W$.

Suppose $\mathcal{M}, x \models \langle\psi\rangle\chi$. Hence, $\mathcal{M}, x \models \psi$ and $\mathcal{M}_{|\psi}, x \models \chi$. Since $\psi \in FOR$, therefore $\{y \in W\colon \mathcal{M}, y \models \psi\} = \{y \in W\colon \mathcal{M}, y \models \tau(\psi)\}$ and $\mathcal{M}_{|\psi} = \mathcal{M}_{|\tau(\psi)}$. Moreover, since $\chi \in FOR$, $\mathcal{M}, x \models \psi$ and $\mathcal{M}_{|\psi}, x \models \chi$, therefore $\mathcal{M}, x \models \tau(\psi)$ and $\mathcal{M}_{|\psi}, x \models \tau(\chi)$. Since $\mathcal{M}_{|\psi} = \mathcal{M}_{|\tau(\psi)}$, therefore $\mathcal{M}_{|\tau(\psi)}, x \models \tau(\chi)$. Thus, $\mathcal{M}, x \models \langle\tau(\psi)\rangle\tau(\chi)$.

Suppose $\mathcal{M}, x \models \langle\tau(\psi)\rangle\tau(\chi)$. Hence, $\mathcal{M}, x \models \tau(\psi)$ and $\mathcal{M}_{|\tau(\psi)}, x \models \tau(\chi)$. Since $\psi \in FOR$, therefore $\{y \in W\colon \mathcal{M}, y \models \psi\} = \{y \in W\colon \mathcal{M}, y \models \tau(\psi)\}$ and $\mathcal{M}_{|\psi} = \mathcal{M}_{|\tau(\psi)}$. Moreover, since $\chi \in FOR$, $\mathcal{M}, x \models \tau(\psi)$ and $\mathcal{M}_{|\tau(\psi)}, x \models \tau(\chi)$, therefore $\mathcal{M}, x \models \psi$ and $\mathcal{M}_{|\tau(\psi)}, x \models \chi$. Since $\mathcal{M}_{|\psi} = \mathcal{M}_{|\tau(\psi)}$, therefore $\mathcal{M}_{|\psi}, x \models \chi$. Thus, $\mathcal{M}, x \models \langle\psi\rangle\chi$.